Math-601D-201: Lecture 21. Pseudo-convex domains and $\bar{\partial}$-equation

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Calculus on (p, q)-forms

$\Omega \subset \mathbb{C}^n$ connected open set.

- (p, q)-forms

$$u = \sum_{|I|=p,|J|=q} u_{I,J}(z) dz^I \wedge d\bar{z}^J$$

- $du = \partial u + \bar{\partial} u$ where ∂u is a $(p+1, q)$ form and $\bar{\partial} u$ is a $(p, q+1)$ form
- $\bar{\partial} u = \sum_{|I|=p,|J|=q} \bar{\partial}(a_{I,J}) \wedge dz^I \wedge d\bar{z}^J$
- $d^2 = 0$ implies $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$

Calculus on (p, q)-forms

$f : \Omega_1 \to \Omega_2$ holomorphic, and ω smooth (p, q)-form in Ω_2. Then

$$f^*(\bar\partial\omega) = \bar\partial(f^*\omega)$$

$\longrightarrow$ the operator $\bar\partial$ can be transported to any complex manifold.

Definition

$\Omega \subset \mathbb{C}^n$.

$$H^{p,q}(\Omega) = \left\{\omega \in \mathcal{C}^\infty_{p,q}(\Omega),\ \ \bar\partial\omega = 0\right\} / \bar\partial\mathcal{C}^\infty_{p,q-1}(\Omega)$$

Resolution of $\bar\partial$ operators on pseudo-convex domains

Theorem

Let $\Omega \subset \mathbb{C}^n$ *be any pseudo-convex domain.*
For any smooth $(p, q+1)$*-forms* f *on* Ω *satisfying* $\bar\partial f = 0$*, there exists a smooth* (p, q)*-form* u *such that* $\bar\partial u = f$*. In other words,*

$$H^{p,q}(\Omega) = 0 \text{ for all } q > 0.$$

$\longrightarrow$ we are going to follow Hörmander's approach based on Hilbert spaces technics

Characterization of pseudo-convex domains

Theorem

$\Omega \subset \mathbb{C}^n$. *The following are equivalent.*

- Ω *is a domain of holomorphy;*
- Ω *is pseudo-convex;*
- $H^{p,q}(\Omega) = 0$ *for all* $q > 0$*;*
- $H^{0,q}(\Omega) = 0$ *for all* $0 < q < n$*.*

Behnke-Thullen's theorem

Theorem

$\Omega_j \subset \Omega_{j+1} \subset \mathbb{C}^n$ pseudo-convex domains.
Then $\bigcup_j \Omega_j$ is pseudo-convex.